

# Schwarzschild and de Sitter solutions from the argument by Lenz and Sommerfeld

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## Abstract

The Lenz-Sommerfeld argument allows an ingenious and simple derivation of the Schwarzschild solution of Einstein equations of general relativity. In this paper, we use the same reasoning to construct the de Sitter line element.

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## I. GENERAL COMMENTS ON GRAVITY

Einstein's first step towards the general theory of relativity was a *gedanken* experiment with an elevator. He considered an observer locked in a box at rest in the Earth's gravitational field performing simple experiments, such as throwing objects and timing the period of a pendulum. This person would see the projectile describing a parabolic trajectory downward [1]

$$y = (\tan \theta_0) x - \left( \frac{g}{2v_0^2 \cos^2 \theta_0} \right) x^2 , \quad (1)$$

where  $g$  is the magnitude of the Earth's gravitational acceleration,  $y$  is the vertical distance,  $x$  is the horizontal range and  $v_0$  is the velocity upon launch at an angle  $\theta_0$  with the horizontal direction. Also, he would verify that the period  $T$  of the simple pendulum measured with the clock could also be obtained by calculating

$$T = 2\pi \sqrt{\frac{L}{g}} ; \quad (2)$$

$L$  is the length of the pendulum. Then Einstein imagined this same box, observer and experiments put in the interstellar space freed from any gravitational influence, accelerated upwards with magnitude  $g$ . As he reported later,<sup>1</sup> Einstein was astonished with the fact that the results of the experiments carried by the observer in this new situation would be exactly the same: the thrown object would still follow a parabolic path – according to Eq. (1) – and the period of the pendulum would not change – Eq. (2) would be valid yet.

Instead, if we consider the elevator free falling in the Earth's gravitational field, the observer (of mass  $m$ , say) would cease to press the floor with a force of magnitude  $mg$ : in the non-inertial reference system attached to the elevator, he is “weightless”, in exactly the same way as the space shuttle astronauts float along with their equipment while they orbit Earth. In the free falling reference system,  $g = 0$  and the parabolic trajectory of ballistic motion degenerates into a straight line, once Eq. (1) reduces to

$$y = (\tan \theta_0) x , \quad (3)$$

Moreover, the period of oscillation of the pendulum becomes infinity:  $T \rightarrow \infty$  as  $g \rightarrow 0$  in accordance with Eq. (2).

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<sup>1</sup> As stated in Ref. [1], Chapter 15, page 430.

The reason for Einstein's amusement is born in a hypothesis hidden in Eq. (2), namely the *equivalence between inertial and gravitational mass*:

$$m_i = m_g . \quad (4)$$

Should the coefficient that measures the response to any kind of force  $m_i$  (the mass in Newton's second law of motion,  $\mathbf{F} = m_i \mathbf{a}$ ) be any different from the inertia coefficient to gravitation  $m_g$  (the mass in the definition of weight,  $\mathbf{W} = m_g \mathbf{g}$ ), the period of a pendulum would be given by

$$T = 2\pi \sqrt{\frac{m_i L}{m_g g}} \quad (5)$$

rather than by Eq. (2). Experiments of extremely high accuracy, such as those by R. Eötvös with the torsion balance, guarantee that Eq. (4) holds up to a precision of 1 part in  $10^{12}$ .

The second move in Einstein's reasoning was to associate gravitation and inertia to world curvature. This way, physics was brought from the flat spacetime structure of Minkowski's line element of special relativity to the curved geometry studied by Gauss, Riemann and Levi-Civita, among others. According to this innovative idea, gravity would not be described by a vector field related to Newton's law,

$$\mathbf{F}_N(r) = -\frac{GmM}{r^2} \hat{r} , \quad (6)$$

for the attractive gravitational force  $\mathbf{F}_N$  acting on bodies of masses  $m$  and  $M$  separated by a distance  $r$  along the direction connecting the centers of mass. In Einstein's theory of gravitation, the field has a tensorial character: it corresponds to a  $4 \times 4$  matrix-like object with ten independent components,  $g_{\mu\nu}$ , defining the arc length  $s$  and infinitesimal distances

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (7)$$

on the curved manifold of experience. Following Mach's ideas, Einstein proposed that the presence of mass sets the stage for the events to take place:  $g_{\mu\nu}$  is a solution of [2]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} . \quad (8)$$

The distribution of mass is modeled by the stress-energy tensor  $T_{\mu\nu}$ ; the Ricci tensor  $R_{\mu\nu}$  and scalar curvature  $R$  are functions of (the derivatives of)  $g_{\mu\nu}$ . The factor  $\kappa$  is basically the Newton's constant  $G$  appearing in Eq. (6) divided by (the fourth power of) the speed of

light  $c$ :  $\kappa = 8\pi G/c^4$ . The cosmological constant  $\Lambda$  was introduced by Einstein so he could derive, out of his theory, a static spherical model for the universe [3], which matched the beliefs of that epoch.

Nowadays,  $\Lambda$  is essential to physical cosmology for the opposite reason, once it allows for the ever expanding cosmos. The so called de Sitter model, named after the man who built it in 1917 [4], is a solution of (8) that predicts an increasingly accelerated recession of the galaxies. The de Sitter interval takes the form

$$ds^2 = \frac{dr^2}{1 - \frac{\Lambda r^2}{3}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - \left(1 - \frac{\Lambda r^2}{3}\right) c^2 dt^2 \quad (9)$$

in the static spherical coordinates  $(r, \theta, \phi)$ . In comoving coordinates, the de Sitter line element is written as

$$ds^2 = a_0 e^{\sqrt{\frac{4\Lambda c^2}{3}}(t-t_0)} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] - c^2 dt^2 \quad (10)$$

where the radial coordinate  $r$  is dimensionless and the physical distance is given by the product of  $r$  by the scale factor  $a(t)$  given by (see e.g. [5]):

$$a(t) = a_0 e^{\sqrt{\frac{\Lambda c^2}{3}}(t-t_0)} \quad (11)$$

( $a_0 = a(t_0)$  is the initial condition set for the scale factor<sup>2</sup>), which is equivalent to say that it is  $a(t)$  that carries the dimension of length. Observing the form of the dependence of  $a$  on  $t$  in Eq. (11), it is clear that the de Sitter solution describes the accelerated expanding universe.

The scale factor (11) can also be determined using the Friedmann equations

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2} \quad (12)$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda c^2}{3} - \frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \quad (13)$$

for  $a(t)$  appearing in a general metric of the form

$$ds^2 = -c^2 dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \quad (14)$$

where  $k$  is a number describing a space section with (i) flat geometry for  $k = 0$ ; (ii) spherical geometry if  $k = +1$ ; and (iii) hyperbolic geometry for  $k = -1$ . In the pair (12-13),  $\rho$  and  $p$

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<sup>2</sup> Actually,  $t_0$  usually represents the present-day value of the cosmic time.

are the mass density and the pressure of the massive content in our model of universe. The dot means derivative with respect to the cosmic time:  $\dot{a} = \frac{da}{dt}$ .

The second Friedmann equation enables us to give an alternative interpretation to the effect of the cosmological constant  $\Lambda$  [5]. For this purpose, consider a region limited by a sphere of physical radius  $a(t)r_0$  (with a constant  $r_0$ ). Then, from Eq. (13),

$$\frac{d^2(ar_0)}{dt^2} = \frac{\Lambda c^2}{3}(ar_0) - \frac{GM}{(ar_0)^2} \quad (15)$$

where we have identified

$$M = \frac{4\pi}{3} \left( \rho + \frac{3p}{c^2} \right) (ar_0)^3, \quad (16)$$

the *total mass* within the sphere: amongst all the energetic density  $(\rho + \frac{3p}{c^2})$  homogeneously and isotropically distributed in the universe, we select the sector inside the volume  $\frac{4\pi}{3}(ar_0)^3$  of the sphere under consideration. Multiplying Eq. (15) by the mass  $m$  of a test particle, we get a force equation

$$F = F_\Lambda + F_N \quad (17)$$

whose second term is the Newtonian law of gravity (6), whereas the first term on the r.h.s,

$$\mathbf{F}_\Lambda(r) = m \frac{\Lambda c^2}{3} r \hat{r}, \quad (18)$$

can be interpreted as a repulsive force due to  $\Lambda$ .<sup>3</sup> In this approach, the force  $\mathbf{F}_\Lambda$  would be the one driving the expansion of the universe. And the recent observations [6, 7] favor this accelerated expanding scenario.

Einstein's theory of gravity is not only able to describe phenomena explained by the Newtonian approach but also successfully addresses the observational puzzle of the perihelion shift of Mercury [8]. It also predicts that light rays coming from distant stars should be deviated from straight trajectories when passing by the Sun [9], a fact that was verified by Eddington [10].

The new theory calls for a philosophical change in the the way we interpret gravity, and it comes along with a new set of sophisticated mathematical tools. Schwarzschild had to

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<sup>3</sup> The interpretation of the repulsive effect of  $\Lambda$  in terms of a force is not de Sitter's contribution, and actually it is not strictly meaningful in the context of general relativity. In Einstein theory, the concept of force is abandoned in favor of the curvature of the spacetime. However, as one comes to  $\mathbf{F}_\Lambda(r)$  by taking the Friedmann equations as the first step of the derivation, the interpretation is at least in agreement with the results from the general theory of gravity.

integrate the set of partial coupled differential equations (8) in order to obtain the line element for the spacetime surrounding a massive body of the kin of a planet or a star (of mass  $M$ ). His solution, found as early as 1916, reads [11]

$$ds^2 = \frac{dr^2}{1 - \frac{2\alpha}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - \left(1 - \frac{2\alpha}{r}\right) c^2 dt^2 \quad (19)$$

with  $\alpha$  standing for the geometrical mass of the source

$$\alpha = \frac{GM}{c^2} . \quad (20)$$

To solve Einstein's equations given a certain distribution of matter and energy can be an extremely laborious task, if possible at all. Lenz developed an alternative method for deriving the Schwarzschild solution. He did not publish his result, as far as these authors are concerned, but he communicated his argument to Sommerfeld in 1944, which reproduced its general lines in his classic treatise on electrodynamics [12]. Lenz's approach combines Newton's gravitation law with Einstein's special theory of relativity to get a non-pseudo-Euclidean spacetime. In this paper we will revise this reasoning (Section II), showing its solid grounds and the efficiency of this approximative technique. We will also extend it to produce the de Sitter solution (Section III).

## II. THE ARGUMENT BY LENZ AND THE SCHWARZSCHILD INTERVAL

As Lenz's reasoning is based on special theory of relativity, it is appropriate to revisit the concepts of proper time and proper length. Special relativity rises from two postulates: (1) the laws of nature are the same for all observers in inertial (non-accelerated) reference frames, and none of them is preferred; (2) the value of the speed of light in vacuum is a constant  $c$  in all inertial reference systems. Hence, two observers in different inertial reference frames will be constrained to measure time intervals and lengths in such a way that  $c$  is the same constant for both of them.

A proper time interval  $\Delta t_0$  is the time lapse between two events at the *same location*, as measured by a stationary clock at that location [1]. The reference frame in which one measures the proper time may be in relative motion with respect to another inertial reference system (the reciprocal velocity being  $v$ , say). In this second reference frame, the events will

occur in *different places*, and the time interval  $\Delta t$  will be

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - (v/c)^2}} . \quad (21)$$

The Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \quad (22)$$

is always greater than 1, once the speed parameter

$$\beta = \frac{v}{c} \quad (23)$$

is less than 1 for any nonzero relative velocity. Therefore,  $\Delta t > \Delta t_0$  and we get the time dilation relation

$$\Delta t = \gamma \Delta t_0 . \quad (24)$$

The length of an object measured in an inertial reference frame in which this object is at rest is the proper length  $l_0$ . In any other reference system in relative motion with the previous one, an observer will measure a contracted length  $l$

$$l = \frac{l_0}{\gamma} . \quad (25)$$

Eq. (25) is a direct consequence of time dilation [1].

With these concepts fresh in our minds, let us go back to Einstein's thought-experiment with the elevator: and so begins the Lenz-Sommerfeld's argument. We will be concerned with the elevator's free fall in a gravitational field (rather than with the part of the *gedanken* experiment when we study the elevator accelerating upwards in the absence of gravitational influence).

Consider a reference frame  $K_\infty$  attached to the elevator. The other reference frame, called  $K$ , will be fixed at the centre of a spherically symmetric source of the gravitational field. This source can be taken as the Sun, of mass  $M$ . The elevator  $K_\infty$  will be falling radially toward  $K$ , which may be regarded as at rest. As we discussed in Section I, such a freely falling system perceives a world free from gravitation and, hence, from curvature. This means that an observer in  $K_\infty$  will measure distances according to the flat Minkowski line element of special relativity

$$ds^2 = dx_\infty^2 + dy_\infty^2 + dz_\infty^2 - c^2 dt_\infty^2 , \quad (26)$$

where  $(x_\infty, y_\infty, z_\infty, t_\infty)$  are the coordinates measured in  $K_\infty$ .

Let  $(r, \theta, \phi, t)$  be the coordinates measured in system  $K$  of the Sun, which is subjected to gravitation. Suppose that the “moving” system  $K_\infty$  arrives at a distance  $r$  from the system  $K$  at “rest” with velocity  $v = \beta c$ , cf. Eq. (23). In addition, the  $x_\infty$ -axis will be taken as the direction of motion: longitudinal, the same as  $r$  direction. This way,  $y_\infty$  and  $z_\infty$  are transversal directions. There will be length contraction along the direction of motion, meaning that the intervals  $dr$  and  $dx_\infty$  will be related by Eq. (25). Two events – such as to turn on a light tube and then turn it off – are measured at the *same place* in the moving reference frame  $K_\infty$  only. Therefore, the proper length is measured by the observer inside the elevator:  $dx_\infty$  is the proper length while  $dr$  is the contracted interval,

$$dr = \frac{dx_\infty}{\gamma} . \quad (27)$$

There is no contraction along the directions  $y_\infty$  and  $z_\infty$  orthogonal to the radial motion. For this reason, these  $K_\infty$ -coordinates will relate to the spatial  $K$ -coordinates through the simple coordinate transformations relating Cartesian coordinates and spherical coordinates:

$$dy_\infty = r d\theta \quad (28)$$

and

$$dz_\infty = r \sin \theta d\phi \quad (29)$$

Moreover, as the proper time interval separates two events happening at the same location, it is measured in the system  $K_\infty$ . Consequently, the dilated time interval will be the one realized in reference frame  $K$ . From Eq. (24), it results:

$$dt = \gamma dt_\infty . \quad (30)$$

By substituting Eqs. (27) to (30) into Minkowski line element (26), one obtains:

$$ds^2 = \frac{dr^2}{(1 - \beta^2)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - c^2 dt^2 (1 - \beta^2) , \quad (31)$$

where we have used Eq. (22). The coordinates in line element (31) are those measured in the system of reference  $K$  attached to the Sun. Notwithstanding, the factor  $\beta$  appearing in (31) is meaningful in connection with the frame fixed in our elevator: only then  $\beta$  can be interpreted as the speed parameter  $\beta = v/c$ , because it is the box  $K_\infty$  that carries



continuously with itself the pseudo-Euclidean metric of special relativity. The meaning of  $\beta$  in the reference frame  $K$  is determined using *conservation of energy*.

In the approach by Lenz and Sommerfeld, the interpretation of gravity as a long range force is combined with the one in which a gravitational field means curvature of space and time. The metric structure in Eq. (26) and (31) is consistent with this last point of view. In considering the energy of the elevator, we will use the Newtonian ideas for gravity when assuming that the elevator bears a *potential energy*  $U(r)$ . We claim that this can be regarded as an adequate first approximation. This potential energy is calculated in the standard way [1] as

$$F(r) = -\frac{dU}{dr} \quad (32)$$

or

$$U(r) = -\int_{r_\infty}^r dr' \hat{r}' \cdot \mathbf{F}(r') , \quad (33)$$

We use  $r'$  within the integration sign to avoid confusion with the limit of integration. (Nevertheless the meaning is clear: either  $r$  or  $r'$  denote the radial direction.) Following Lenz, Sommerfeld substitutes the force  $\mathbf{F}(r)$  that enters Eq. (33) by the Newton's law of gravitation, Eq. (6). Integrating, it results:

$$U(r) = -\frac{GmM}{r} - U(r_\infty) \quad (34)$$

where

$$U(r_\infty) = -\frac{GmM}{r_\infty} . \quad (35)$$

As the potential is always defined up to a constant, one may set its zero level at  $r = r_\infty$ :

$$U(r_\infty) = 0 . \quad (36)$$

This is easily justified: by taking  $r_\infty \rightarrow \infty$ , it follows  $U_N(r_\infty) = 0$ , so that

$$U_N(r) = -\frac{GmM}{r} . \quad (37)$$

(The index  $N$  of  $U(r)$  stands for *Newtonian* potential.)

The expression for the relativistic *kinetic energy* of the elevator is:

$$T = m_0 c^2 (\gamma - 1) = (m - m_0) c^2 \quad (38)$$

where its mass  $m$ ,

$$m = \frac{m_0}{\sqrt{1 - \beta^2}} \quad (39)$$

is given in terms of the rest mass  $m_0$  [13]. Notice that when the elevator is at rest  $v = 0$ ,  $m = m_0$  and consequently  $T = 0$ . This situation occurs as initial condition since it is assumed that the elevator falls from rest at  $r = r_\infty$ :

$$T(r_\infty) = 0 . \quad (40)$$

The conservation of the *total energy*

$$E = T + U \quad (41)$$

gives:

$$E(r) = E(r_\infty) \quad (42)$$

However, in face of Eqs. (36) and (40),  $E(r_\infty) = 0$ , and we are left with

$$(m - m_0)c^2 - \frac{GmM}{r} = 0 , \quad (43)$$

where we have used Eqs. (34) and (38) for the potential and kinetic energy of the elevator at a distance  $r$  from the Sun as measured by an observer on  $K$ .

Dividing Eq. (43) by the mass  $m$  of the test particle,

$$\left(1 - \frac{m_0}{m}\right) - \frac{GM}{c^2} \frac{1}{r} = 0 , \quad (44)$$

and substituting the ratio of the masses according to Eq. (39):

$$\left(1 - \sqrt{1 - \beta^2}\right) - \frac{\alpha}{r} = 0 . \quad (45)$$

$\alpha$  is the definition of the geometrical mass – Eq. (20) – appearing in the Schwarzschild line element. For our Sun,  $\alpha \simeq 1.5$  km [14]. From Eq. (45):

$$\sqrt{1 - \beta^2} = 1 - \frac{\alpha}{r} ,$$

i.e.,

$$(1 - \beta^2) = \left(1 - \frac{\alpha}{r}\right)^2 = \left[1 - \frac{2\alpha}{r} + \left(\frac{\alpha}{r}\right)^2\right] . \quad (46)$$

It is come the time for an *approximation*. We will consider that the test particle occupies positions at great distances from the source, so that

$$\frac{\alpha}{r} \ll 1 . \quad (47)$$

Under the approximation (47), the term  $\left(\frac{\alpha}{r}\right)^2$  of Eq. (46) is a second order one in  $\frac{\alpha}{r}$ . Hence it is negligible, leading to:

$$(1 - \beta^2) \simeq \left(1 - \frac{2\alpha}{r}\right). \quad (48)$$

This determines the meaning of  $\beta$  in the system  $K$ .

Substituting Eq. (48) into our expression for the line element, Eq. (31), we find:

$$ds^2 = \frac{dr}{\left(1 - \frac{2\alpha}{r}\right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - c^2 dt^2 \left(1 - \frac{2\alpha}{r}\right). \quad (49)$$

This is precisely the Schwarzschild solution (19) of Einstein's equation. The power of Lenz-Sommerfeld's argument is thus unveiled: the derivation presented here could only guarantee an approximated result; however it coincides with the exact solution of the ten non-linear coupled partial differential equations (8) in the presence of a massive source.

### III. THE DE SITTER SOLUTION VIA LENZ'S ARGUMENT

Our additional step in this paper is to add to  $\mathbf{F}(r)$  the contribution coming from the de Sitter's force, Eq. (18). Therefore, we admit that the test particle of mass  $m$  (elevator) is subjected to the linear repulsive force  $\mathbf{F}_\Lambda(r)$  coming as an effect of a non-null cosmological constant  $\Lambda$ , and which leads to a potential

$$U(r) = -m \frac{\Lambda c^2}{6} r^2 - U(r_\infty) \quad (50)$$

where

$$U(r_\infty) = -m \frac{\Lambda c^2}{6} r_\infty^2. \quad (51)$$

after integration of (33). Again, the additive constant in the expression for the potential energy can be set to zero, but the explanation for taking

$$U(r_\infty) = 0 \quad (52)$$

in this case associated with  $\Lambda$  is different from the Schwarzschild case. The pathology here comes from the fact that  $r_\infty = \infty$  would imply  $U_\Lambda(r_\infty) \rightarrow \infty$ . This apparent problem is solved by taking  $r_\infty = 0$  which automatically renders the potential null  $U_\Lambda(0) = 0$  by Eq. (51). Thus, what we are really doing is assuming the level zero for the potential at the “origin” of the de Sitter solution rather than at its “bondary”. Instead of calculating the

potential as the energy necessary to bring a test particle from the infinity to the position  $r$ , we define  $U_\Lambda(r)$  by considering the transport of the test particle from  $r = 0$  to an arbitrary position  $r$ . Therefore,

$$U_\Lambda(r) = -m \frac{\Lambda c^2}{6} r^2$$

In the de Sitter case, the equation for the relativistic *kinetic energy* of the elevator is still Eq. (38). We also take the elevator at rest at the de Sitter radius  $r_\infty = L$ , leading once more to the constraint  $T(r_\infty) = 0$ . This initial condition is in agreement with the predictions of general relativity: on Tolman's book [3], one can read about the motion of a test particle in the de Sitter universe and understand, based on the integration of the geodesic equations, that it takes an infinity time for the particle traveling toward the boundary to reach the horizon, where its velocity would be zero.

Imposing conservation of the *total energy* for the elevator's free fall in de Sitter universe, gives now

$$(m - m_0) c^2 - m \frac{\Lambda c^2}{6} r^2 = 0 . \quad (53)$$

If we proceed as in Section II, and divide Eq. (53) by the mass  $m$  of the test particle, we come to

$$\sqrt{1 - \beta^2} = 1 - \frac{\Lambda}{6} r^2 ,$$

i.e.,

$$(1 - \beta^2) = \left(1 - \frac{\Lambda}{6} r^2\right)^2 = \left[1 - \frac{\Lambda}{3} r^2 + \left(\frac{\Lambda}{6} r^2\right)^2\right] . \quad (54)$$

At this point we will perform an *approximation*, assuming that the value of the cosmological constant is small

$$\Lambda r^2 \ll 1 , \quad (55)$$

which is actually true in view of the constraints imposed by the dynamics of our solar system and the standard cosmological model. In fact, one can easily estimates  $\Lambda \simeq 1.38 \times 10^{-52} \text{ m}^{-2}$  using recent astrophysical data available at the Particle Data Group website [15]. The effects of the cosmological constant are only important in a cosmological perspective, when the repulsive force (18) becomes significantly effective (because  $r$  is very large). Under the approximation (55), the term  $\left(\frac{\Lambda}{6} r^2\right)^2$  of Eq. (54) is of second order in  $\Lambda r^2$ :

$$(1 - \beta^2) \simeq \left(1 - \frac{\Lambda}{3} r^2\right) . \quad (56)$$

Inserting Eq. (56) into Eq. (31), renders the line element

$$ds^2 = \frac{dr^2}{\left(1 - \frac{\Lambda}{3}r^2\right)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - c^2 dt^2 \left(1 - \frac{\Lambda}{3}r^2\right), \quad (57)$$

the de Sitter solution in its static form (9). We just got another standard exact solution of Einstein's equation from Lenz-Sommerfeld's approximative procedure.

#### IV. FINAL COMMENTS

In this paper we reviewed the argument by Lenz and Sommerfeld leading to the Schwarzschild solution of general relativity using only concepts from the special theory of relativity and the Newtonian theory of gravity.

We showed that it is also possible to employ Lenz-Sommerfeld's technique to obtain a solution that includes the cosmological constant. So we built the de Sitter solution. The later is of great importance for modern cosmology once it allows for an accelerated expansion of the universe, something that is favorable to the recent data. The derivation of de Sitter solution from Lenz's argument was ultimately possible due to the universal character of the cosmological constant: it responds to gravitation in the same way as all masses do.

The spirit of the paper is to point out a non-standard procedure of deriving the classical solutions of general relativity. A natural following step would be to apply Lenz-Sommerfeld's reasoning to calculate the metric of a slowly rotating massive source, known as Lenz-Thirring line element. This is an investigation under development by the authors.

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